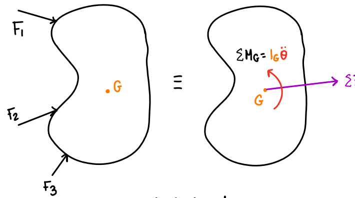
Kinematics and Dynamics of Rigid Bodies

Translation & Rotation

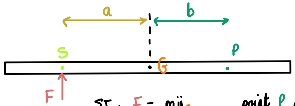


If rigid body has no axis of rotation, it will rotate around its centre of mass under applied couple.

∴ consists of linear acceleration a of its CoM and angular acceleration $\ddot{\theta}$ around its CoM

Centre of Percussion

Point S that can be struck without causing acc. at another point P



ΣF_y: $F = m\ddot{y}_G$

ΣM_G: $-Fa = I_G\ddot{\theta}$

$\ddot{y}(x) = \ddot{y}_G + \ddot{\theta}x$

point P at $x = b$ has instantaneous acc. of 0:

$\frac{F}{m} - b\frac{Fa}{I_G} = 0$

$\therefore ab = \frac{I_G}{m} = k_G^2$

Radius of Gyration

alternative description for I_0

or consider as slender strip with equivalent mass

point mass m placed distance k from axis of rotation produces equivalent I_0

$I = \int r^2 dm = mk^2$

$\therefore k = \sqrt{\frac{I}{m}}$

Captures distribution of mass in a rigid body

e.g. solid disk $K_G \approx \frac{R}{\sqrt{2}}$

spoked wheel $K_G \approx R$

Moment of Inertia

For Rod:

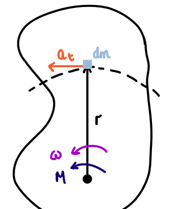
$I_G = \frac{mL^2}{12}$

$I_0 = \frac{mL^2}{3}$

For Disk | Wheel

$I_G = \frac{mR^2}{2}$

Derivation:



$\Sigma M = \int (a_t\, dm)\, r = \ddot{\theta}\boxed{\int r^2 dm}$

$= I_0\ddot{\theta}$

$\Sigma M_0 = I_0\ddot{\theta}$

$\boxed{I_0 = \int r^2 dm}$

analogous to rotational $\Sigma F = ma$

ability to resist changes in angular velocity

∴ large I_0 requires large M_0 to rotate

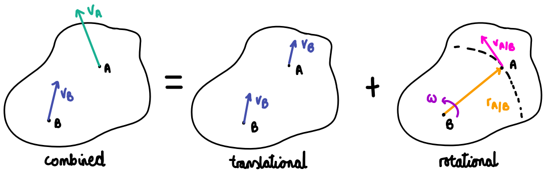
also depends on distribution of mass around axis of rotation O.

Parallel Axis Theorem

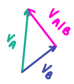
- moment of inertia about any parallel axis can be related to moment of inertia about centre of mass and distance to pivot point

$I_0 = I_G + d^2 m$

Relative Motion



$V_A = V_B + V_{A/B}$



$V_{A/B} = \omega \times r_{A/B}$

For either reference point A and B, the velocities must be identical ∴ same motion

$V_B = V_A + V_{B/A}$ $= -V_{A/B}$ A/B = 'A from B'

angular velocity, $\omega = \frac{d\theta}{dt} = \dot{\theta}$

angular acceleration, $\alpha = \frac{d\omega}{dt} = \ddot{\theta}$

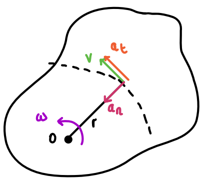
→ $\alpha\, d\theta = \omega\, d\omega$

Fixed point of rotation O and constant r:

$V = \omega r$

$a_n = \omega^2 r = \frac{v^2}{r}$

$a_t = \alpha r$



In vector form:

→ allowing 3D extension

$V = \omega \times r$

$a_n = \omega \times (\omega \times r)$

$a_t = \alpha \times r$

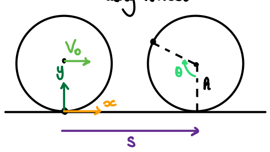
Gears and Belts:

$V = r_1\omega_1 = r_2\omega_2$

Gear Teeth: $N_1\omega_1 = N_2\omega_2$

$a_t = r_1\alpha_1 = r_2\alpha_2$

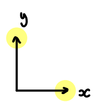
Rolling Wheel



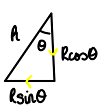
$S = \theta R$

$V_0 = \omega R$

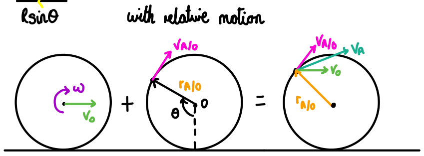
$a_0 = \alpha R$



$x = R(\theta - \sin\theta)$
$y = R(1 - \cos\theta)$ } take differentials for $\dot{x}$, $\ddot{x}$ etc.



with relative motion



$R_{A/0} = \boxed{-R\sin\theta - R\cos\theta}$

$V_{A/0} = \omega \times r_{A/0} = \boxed{-R\omega\cos\theta\, i + R\omega\sin\theta\, j}$

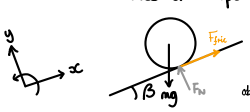
$V_A = V_0 + V_{A/0} = \omega R(1 - \cos\theta)\underline{i} + \omega R\sin\theta\, \underline{j}$

$a_A = a_0 + a_{A/0}$

$= \alpha R + \omega \times (\omega \times r) + \alpha \times r$

$a_A = (\alpha R - \alpha R\cos\theta + \omega^2 R\sin\theta)\underline{i} + (\omega^2 R\cos\theta + \alpha R\sin\theta)\underline{j}$

Wheel on Slope



if $\mu F_N = F_{fric}$ object is about to slide → impending slip

will roll without slipping when

$|F_{fric}| > \mu F_N$ or $\mu < \frac{\tan\beta}{3}$

$\ddot{x} = -\frac{g\sin\beta}{1 + (K_G/R)^2}$

↳ rolling slows down motion (due to I_0) by a factor of $1 + (K_G/R)^2$

Instantaneous Point of Zero Velocity

absolute $v = 0$

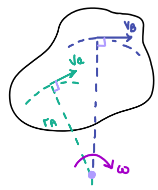
→ all points appear to rotate around P

at point P:

$V = 0$

$a \neq 0$ ∴ point will move in space

- may lie in body or outside
- not fixed in space



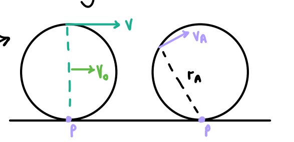
Velocities tangent to their local circular motion

→ take normals to local v's

→ intersection of normals = P

$\omega = \frac{V_A}{r_A}$ ∴ $\boxed{V_x = \omega r_{x/P}}$

Rolling Wheel with P

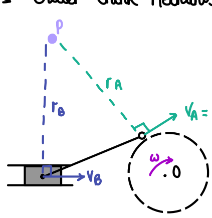


$V = \omega 2R$

$V_A = \omega r_{A/P}$

↳ Velocity of any point on wheel

Slider-Crank Mechanism



$\omega_{AB} = \frac{V_A}{r_A} = \frac{\omega R}{r_A}$

used to find absolute velocity of slider → $V_B = \omega_{AB} r_B$

Absolute Motion Solving:

1. Write equation relating an angle that changes to a length that changes
2. Take derivatives for v and a or ω and α